

# Tropical Nets and Their Role in Kleene Stars of Tropical Matrices

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## Abstract

We consider the classical max-plus tropical semiring  $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$  and its extended version  $\overline{\mathbb{T}} = \mathbb{R} \cup \mathbb{R}^\nu \cup \{-\infty\}$  [1, 3], endowed with their natural total orders  $\leq$  and  $\preceq$  respectively. In this framework, we introduce a weighted digraph on the non-semiring structure  $\overline{N} := \{t \in \overline{\mathbb{T}} : t \preceq 0^\nu\}$ , called the *tropical net*. A tropical net with  $n$  vertices  $v_1, \dots, v_n$  is a weighted digraph in which each vertex  $v_i$  has a self loop of weight  $w_{ii} = 0^\nu$  and for every pair  $p \neq q$ , there is exactly one directed edge  $e_{pq}$  from  $v_p$  to  $v_q$  with weight  $w_{pq} \in \overline{N}$ . This digraph is designed as an analogue of the Q-net [4], defined over the semiring  $N := \{t \in \mathbb{T} : t \leq 0\}$  by Yu et al. [5]. We undertake a systematic study of the tropical net, focusing on its connectivity, path behavior, and decomposition into strongly connected components. Central to this analysis is the associated *critical digraph*, which captures the critical cycles (cycles with weight  $0^\nu$ ) that determine the overall structure of the tropical net. The presence of ghost elements in  $\overline{\mathbb{T}}$  (elements of  $\mathbb{R}^\nu$ ) introduces new features in the interaction between order and connectivity, leading to refined structural properties not observed in the classical tropical case.

These graph-theoretic considerations are then employed to examine weighted digraphs of the *tropical matrices* (matrices of  $M_n(\overline{\mathbb{T}})$ ) in  $M_n^{\overline{N}} := \{(a_{ij})_{n \times n} \in M_n(\overline{N}) : a_{ii} = 0^\nu \text{ and } a_{ij} \preceq 0^\nu \text{ for all } i, j\}$ . In particular, the tropical net and its critical digraph provide a natural mechanism for identifying and analyzing the cycles that ultimately determine the *Kleene stars* [2] of the matrices in  $M_n^{\overline{N}}$ . This approach yields a unified and combinatorial procedure for computing Kleene stars, extending classical ideas to the enriched tropical framework. In spite of lacking the semiring structure of  $\overline{N}$ , most of the similar properties of  $M_n^N = \{(a_{ij})_{n \times n} \in M_n(\mathbb{T}) : a_{ii} = 0 \text{ and } a_{ij} \leq 0 \text{ for all } i, j\}$  in [5] can be achieved by  $M_n^{\overline{N}}$ .

Overall, the development of the tropical net and its critical digraph yields a systematic, graph-theoretic method for analyzing matrices in  $M_n^{\overline{N}}$ , culminating in a general procedure for computing their Kleene stars.

## References

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\*\*I request that my paper be considered for the Best Paper Presentation Award.